

# String Theory Duals for Mass-deformed $SO(N)$ and $USp(2N)$ $\mathcal{N} = 4$ SYM Theories

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## Abstract

We generalize the results of Polchinski and Strassler regarding the  $\mathcal{N} = 1$ -preserving mass deformation of  $\mathcal{N} = 4$   $SU(N)$  SYM theories and its string theory dual to  $SO(N)$  and  $USp(2N)$  gauge groups. The string theory duals involve 5-branes wrapped on  $\mathbf{RP}^2$ . In order to match with the field theory classification of vacua, the 3-brane charge carried by such 5-branes must be shifted by a half, and this follows from a conjectured generalization of results of Freed and Witten. Our results provide an elegant physical picture for the classification of classically massive vacua in the mass-deformed  $\mathcal{N} = 4$  theories.

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## 1. Introduction

The AdS/CFT correspondence [1,2,3] (see [4] for a review) is conjectured to be an exact duality between certain field theories and certain compactifications of string/M theory. In particular, the  $\mathcal{N} = 4$  four dimensional SYM theory with  $SU(N)$  gauge group is dual to type IIB string theory on  $AdS_5 \times \mathbf{S}^5$ , and the  $\mathcal{N} = 4$  theories with  $SO(N)$  and  $USp(2N)$  gauge groups are dual to type IIB string theory on  $AdS_5 \times \mathbf{RP}^5$  [5].

The correspondence was originally stated for conformal field theories, but it can easily be generalized also to relevant deformations of the conformal field theories, which are realized as solutions of string/M theory with particular boundary conditions depending on the deformation parameter. Some string theory solutions corresponding to the mass deformation of the  $SU(N)$   $\mathcal{N} = 4$  theory were recently discussed in [6]<sup>3</sup>. The vacua of the mass-deformed theory were described in terms of configurations including 5-branes of various types wrapped on  $\mathbb{R}^4 \times \mathbf{S}^2$ . In particular, for the mass deformation which preserves  $\mathcal{N} = 1$  supersymmetry (which was called the  $\mathcal{N} = 1^*$  theory in [6]), a one-to-one mapping was found between classical vacua of the field theory and configurations involving purely D5-branes (though the supergravity approximation is only good for some of these configurations, and others may be better described in terms of other types of branes).

In this paper we wish to generalize the results of [6] to the case of the  $\mathcal{N} = 1^*$   $SO(N)$  and  $USp(2N)$  gauge theories. We will find that again many of the classical vacua of the field theory have a description in string theory in terms of configurations of D5-branes. In particular, all the classically massive vacua (classified in [8]) have such a description. This gives a nice physical realization of the mathematical results of [8] (a different realization of the same results was found in [9]). The 5-branes in our configurations will be wrapped around  $\mathbb{R}^4 \times \mathbf{RP}^2$ , and in order to match with the classical field theory results we will have to use a conjectured generalization of the results of [10] concerning charge quantization to orientifolded backgrounds. This leads to a half-integral shift in the 3-brane charge carried by 5-branes wrapped on  $\mathbf{RP}^2$ , such that in the  $USp(2N)$  case such D5-branes carry integer 3-brane charges, while in the  $SO(N)$  case the 3-brane charge is an integer plus a half.

The string theory solutions we find are almost identical to those found in [6], and we will focus here just on the aspects which are different in the  $SO(N)$  and  $USp(2N)$  theories. One such aspect is the fact that not all the classical vacua may be mapped to D5-brane configurations. It would be interesting to check if the other vacua have a dual description in terms of some other configuration of 5-branes.

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<sup>3</sup> Very recently, these results were generalized to deformations of the  $d = 3$   $\mathcal{N} = 8$  SCFTs [7].

## 2. The Vacua of the Mass-deformed $\mathcal{N} = 4$ SYM Theory

The  $\mathcal{N} = 4$  SYM theory may be described as an  $\mathcal{N} = 1$  gauge theory with three chiral superfields  $\Phi_i$  ( $i = 1, 2, 3$ ) in the adjoint representation, and a superpotential of the form

$$W = \frac{2\sqrt{2}}{g_{YM}^2} \text{tr}([\Phi_1, \Phi_2]\Phi_3). \quad (2.1)$$

There is one possible relevant deformation of this theory which preserves  $\mathcal{N} = 1$  supersymmetry, which is a superpotential of the form

$$\Delta W = \frac{1}{g_{YM}^2} (m_1 \text{tr}(\Phi_1^2) + m_2 \text{tr}(\Phi_2^2) + m_3 \text{tr}(\Phi_3^2)). \quad (2.2)$$

We will take the masses to be equal,  $m_1 = m_2 = m_3 = m$ . It is straightforward to generalize our results to the case of non-equal masses.

The classical vacua of the mass-deformed  $\mathcal{N} = 1^*$  field theory are solutions to the F-term and D-term equations,

$$\begin{aligned} [\Phi_i, \Phi_j] &= -\frac{m}{\sqrt{2}} \epsilon_{ijk} \Phi_k \\ \sum_{i=1}^3 [\Phi_i, \Phi_i^\dagger] &= 0, \end{aligned} \quad (2.3)$$

up to gauge identifications. Setting the D-terms to zero and dividing by the gauge group is the same as dividing by the complexified gauge group  $G_C$ . Up to a constant, the F-terms give precisely the commutation relation of the  $SU(2)$  Lie algebra. Thus, as discussed in [11,12], the classical vacua are in one to one correspondence with complex conjugacy classes of homomorphisms of the  $SU(2)$  Lie algebra to that of the gauge group  $G$ . Since such homomorphisms have no infinitesimal deformations, the chiral superfields are all massive in such vacua. The homomorphism generally breaks the gauge group  $G$  to a subgroup  $H$ , and classically the gauge bosons of  $H$  will be massless. Classically massive vacua correspond to homomorphisms which break the gauge group  $G$  completely (or, at most, preserve a discrete subgroup).

For  $G = SU(N)$ , such homomorphisms are equivalent to  $N$  dimensional representations of  $SU(2)$ . Thus, there is one classical vacuum for every partition of  $N$ , of the form  $N = \sum_{i=1}^r n_i$  with positive integers  $n_i$ . The  $N \times N$  matrices corresponding to a particular partition may be written in a block diagonal form, with blocks of size  $n_i \times n_i$  corresponding to the  $n_i$ -dimensional irreducible representation of  $SU(2)$ . The gauge group is completely

broken in the vacuum corresponding to the  $N$  dimensional representation, while in other vacua there is (classically) an unbroken gauge group. If we denote the number of times the  $d$ -dimensional representation appears in the partition by  $k_d$ , the unbroken gauge group is  $(\prod_{d=1}^N U(k_d))/U(1)$ ; the non-Abelian factors in this gauge group rotate blocks which have the same size. Vacua containing unbroken non-Abelian gauge groups are expected to split into several different vacua in the quantum theory. If there are no unbroken Abelian gauge groups such vacua will have a mass gap.

For other gauge groups, we are not aware of a complete classification of the possible homomorphisms. However, the homomorphisms which completely break the gauge group (up to possible discrete factors) were classified in [8]. For  $G = SO(N)$  one such classically massive vacuum was found for every partition of  $N$  into distinct odd integers, while for  $G = USp(2N)$  one such vacuum was found for every partition of  $2N$  into distinct even integers<sup>4</sup>. As above, these partitions may also be mapped into a block-diagonal form [8]. Partitions in which the (odd or even) integers are not all distinct may similarly be mapped to field theory vacua corresponding to block-diagonal matrices, in which the gauge group is not completely broken; if there are  $k$  blocks of the same size, the unbroken gauge group includes an  $SO(k)$  factor. Again, vacua with unbroken non-Abelian gauge groups will split in the full quantum theory. The vacua of the quantum theory (for any gauge group) may be found by a generalization of the analysis of [13].

Unlike the  $SU(N)$  case, it seems that for  $SO(N)$  and  $USp(2N)$  gauge groups not all the solutions to (2.3) are given by the block-diagonal solutions described above. In the  $USp(2N)$  case, even the trivial solution  $\Phi_i = 0$  is not included in our classification above<sup>5</sup>. Another example is  $SO(6) \simeq SU(4)$  which has a solution breaking the gauge group to  $SU(2) \times U(1)$  which is not in our classification. We will not attempt a complete classification of the solutions to (2.3) here.

### 3. The String Theory Description of the Vacua

The  $\mathcal{N} = 4$  SYM theory with gauge group  $G = SU(N)$  is believed to be dual to type IIB string theory compactified on  $AdS_5 \times \mathbf{S}^5$  with  $N$  units of 5-form flux on the  $\mathbf{S}^5$  [1,2,3].

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<sup>4</sup> This is related to the fact that the odd-dimensional representations of  $SU(2)$  are real, while the even-dimensional representations are pseudo-real.

<sup>5</sup> As we will see in the next section, this means that it is not dual to a background involving purely D5-branes.

Similarly, the theories with gauge groups  $SO(N)$  and  $USp(2N)$  are believed to be dual to compactifications of type IIB string theory on  $AdS_5 \times \mathbf{RP}^5$ , in which non-orientable string worldsheets wrapping the non-trivial  $\mathbf{RP}^2 \subset \mathbf{RP}^5$  are allowed. The  $SO(N)$  and  $USp(2N)$  cases are distinguished by discrete torsion. The RR and NS-NS 3-form fields are twisted by the orientifolding, therefore their cohomology is classified by  $H^3(\mathbf{RP}^5, \tilde{\mathbb{Z}}) = \mathbb{Z}_2$ , as explained in [5]. Thus, for each 3-form field there are two choices of the discrete torsion, leading to four different theories. The theory with no torsion is  $SL(2, \mathbb{Z})$  invariant and is dual to the  $SO(2N)$  gauge theory. The theory with only RR torsion is dual to an  $SO(2N + 1)$  gauge theory, while the two theories with NS-NS torsion are dual to two  $USp(2N)$  theories which differ in their soliton spectrum [14].

The AdS/CFT correspondence may be extended to include relevant deformations of the conformal field theory, which are mapped to string theory in asymptotically AdS backgrounds with particular boundary conditions corresponding to the deformation. In particular, the mass deformation we analyzed above (for equal masses) corresponds to a background including 3-form field strengths (as reviewed in section 4.3 of [4]). We expect to have a string theory solution with the appropriate boundary conditions corresponding to every vacuum of the quantum field theory after the mass deformation.

Recently, Polchinski and Strassler [6] gave a beautiful physical picture of the vacua of the mass-deformed  $SU(N)$  theory, in the limit where supergravity is a good approximation. They noted, following [15], that  $n$  3-branes carrying 3 scalar fields whose VEVs are an  $n$ -dimensional representation of  $SU(2)$  (as described above) may be thought of as a D5-brane wrapped on an  $\mathbf{S}^2$  (in the directions corresponding to the 3 scalar fields), carrying  $n$  units of D3-brane charge coming from a  $U(1)$  gauge field carrying  $n$  units of flux on the  $\mathbf{S}^2$ . Thus, they interpreted the classical field theory vacua we described above, corresponding to a partition  $N = \sum_{i=1}^r n_i$ , as being dual to string theory configurations including  $r$  D5-branes, each carrying  $n_i$  units of 3-brane charge. The geometry of the 5-branes is  $\mathbb{R}^4 \times \mathbf{S}^2$ , where the  $\mathbf{S}^2$  is an equator in the  $\mathbf{S}^5$  of  $AdS_5 \times \mathbf{S}^5$ , and the  $\mathbb{R}^4$  corresponds to a constant radial position in  $AdS_5$  (in Poincaré coordinates).

Polchinski and Strassler constructed approximate string theory solutions corresponding to these vacua [6], and found that the  $i$ 'th D5-brane sits at a radial position in AdS space which is proportional to  $n_i$ . This reproduces elegantly the classically unbroken gauge group described above, which is just the product of the gauge groups on the different D5-branes (which is enhanced to a non-Abelian group when the D5-branes overlap), up to an overall  $U(1)$  factor which may be gauged away [6]. The supergravity approximation used

in [6] is only valid if all the  $n_i$  are much bigger than  $\sqrt{g_s N} \gg 1$ ; if this does not hold there are sometimes alternative descriptions of the same vacua in terms of other types of 5-branes [6]. We will ignore this issue here and assume that in some sense the D5-brane vacua still exist even if the supergravity approximation breaks down.

Most of the results of [6] apply also to the  $SO(N)$  and  $USp(2N)$  theories. The supergravity fields on  $AdS_5 \times \mathbf{RP}^5$  are a  $\mathbb{Z}_2$  projection of the fields on  $AdS_5 \times \mathbf{S}^5$ , and this projection preserves the fields that are excited in the solutions of [6]. Hence, in the supergravity approximation we can find the same solutions involving 5-brane shells as in the  $SU(N)$  case of [6]. The results of [6] concerning confinement, screening, baryons, condensates, domain walls, instantons, flux tubes, and so on, have obvious generalizations to the  $SO(N)$  and  $USp(2N)$  cases. The 5-branes now wrap an  $\mathbf{RP}^2$  which is an equator of  $\mathbf{RP}^5$  instead of wrapping an  $\mathbf{S}^2$ . As discussed in [5], such a wrapping is consistent with the orientifold projection since it inverts the 3-form field coupling to the 5-brane charge.

The main difference between wrapping 5-branes on  $\mathbf{S}^2$  and on  $\mathbf{RP}^2$  is in the allowed amounts of 3-brane charge the 5-brane can carry. The 3-brane charge arises from a term in the effective action of the D5-brane, of the form

$$\int d^6x \, C^{(4)} \wedge (F - B_{NS}). \quad (3.1)$$

Thus, the 3-brane charge is the integral of  $(F - B_{NS})$  over the compact 2-cycle. For an  $\mathbf{S}^2$  in  $AdS_5 \times \mathbf{S}^5$ , this integral gives an integer 3-brane charge by the usual quantization condition. However, in general backgrounds an analysis of worldsheet global anomalies can lead to a shift in this quantization [10]. The quantization in orientable backgrounds, for a D-brane worldvolume of the form  $\mathbb{R}^{p-1} \times Q$  (with a 2-cycle  $Q$ ) was found to be

$$\int_Q \frac{F}{2\pi} = \frac{w_2(Q)}{2} \pmod{1}, \quad (3.2)$$

where  $w_2(Q)$  is the second Stiefel-Whitney class, normalized to take values in  $\{0, 1\}$ . Thus, if  $Q$  has a non-trivial  $w_2$ , the contribution from the gauge field to the 3-brane charge will be an integer plus a half. The results of [10] have not yet been generalized to include non-orientable backgrounds. However, it seems that the formula (3.2) still holds in these backgrounds [5,10], even though  $F$  has to be generalized to a twisted gauge bundle if  $Q$  is non-orientable. We will conjecture that this formula holds for  $Q = \mathbf{RP}^2$ , and the matching with the gauge theory results described in the previous section will give strong evidence for this.

For our case of a 5-brane on  $\mathbb{R}^4 \times \mathbf{RP}^2$ , the second Stiefel-Whitney class is non-trivial, since for 2-cycles  $w_2(Q) = e(Q) \pmod{2}$  where  $e(Q)$  is the Euler class, which equals one for  $\mathbf{RP}^2$ . Thus, the gauge field contribution to the 3-brane charge is an integer plus one half in this case. An additional contribution can come from the NS-NS 2-form field  $B_{NS}$ , integrated over  $\mathbf{RP}^2$  ( $B_{NS}$  is twisted under the orientifolding so we can integrate it over  $\mathbf{RP}^2$ ). As discussed in [5], this integral is exactly the discrete torsion of the NS-NS 3-form field strength, which is classified by  $H^3(\mathbf{RP}^5, \tilde{\mathbb{Z}}) = \mathbb{Z}_2$ . As discussed above, for the backgrounds corresponding to  $SO(2N)$  and  $SO(2N+1)$  gauge groups, this discrete torsion vanishes, while for  $USp(2N)$  it gives

$$\int_{\mathbf{RP}^2} \frac{B_{NS}}{2\pi} = \frac{1}{2} \pmod{1}. \quad (3.3)$$

Thus, we find that for  $SO(2N)$  and  $SO(2N+1)$  theories, the 5-branes carry 3-brane charges in  $\mathbb{Z} + \frac{1}{2}$ , while for  $USp(2N)$  groups they carry integer 3-brane charges.

In the  $SU(N)$  case, we mapped the classical field theory vacua to string theory configurations where all of the 3-brane flux was carried by D5-branes, leaving flat space at the origin [6]. In our case, the total 3-brane flux cannot all be carried by D5-branes. One way to see this is from the fact that the total flux is not an integer, or even a half-integer, but rather is given by

$$\int_{\mathbf{RP}^5} \frac{G^{(5)}}{2\pi} = \begin{cases} N - \frac{1}{4} & \text{for } SO(2N), \\ N + \frac{1}{4} & \text{for } SO(2N+1), \\ N + \frac{1}{4} & \text{for } USp(2N); \end{cases} \quad (3.4)$$

these charges may be deduced from the charges of D3-branes stuck on orientifold planes before taking the near-horizon limit. Thus, at the origin we always remain with some charge that must be carried by an orientifold. Equivalently, it is clear geometrically that since the transverse space is orientifolded, we must remain with an orientifold at the origin and not just with flat ten dimensional space<sup>6</sup>.

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<sup>6</sup> In the  $USp(2N)$  cases, the D5-branes we added do not change the discrete torsion, so the orientifold we remain with at the origin carries the same discrete torsion as the original background. In the  $SO(N)$  cases, each D5-brane is actually a domain wall along which the RR discrete torsion changes [5], so we could end up at the origin with either sign for this discrete torsion. The orientifold with the RR discrete torsion may be viewed as the orientifold without the torsion with a half D3-brane stuck on top of it. It seems that such a half D3-brane may be identified with a one dimensional block in the scalar matrices (recall that for small blocks the description in terms of D5-branes breaks down). With this identification it does not matter which type of orientifold we end up with, and we will assume below that we end up with no discrete torsion.

Now, we can list the string theory vacua where the 3-brane flux is completely carried by D5-branes (except for the contribution of the remaining orientifold), as in the vacua of [6] described above. For  $SO(N)$  gauge groups this charge is  $N/2$ , and this should be reproduced as the sum of contributions in  $\mathbb{Z} + \frac{1}{2}$ ; hence, we find one such vacuum for every partition of  $N$  into odd integers. For  $USp(2N)$  gauge groups, this charge is  $N$ , which should arise as the sum of contributions in  $\mathbb{Z}$ ; thus, we find one vacuum for every partition of  $N$ .

These results agree precisely with the gauge theory results described above, if a single D5-brane wrapped around  $\mathbf{RP}^2$  carries no gauge group while multiple D5-branes carry some non-trivial gauge group; then, we would find no remaining gauge groups exactly when the partitions involve distinct integers. This fact is indeed true, because the orientifolding projects out part of the four dimensional gauge fields living on the D5-branes wrapped on  $\mathbf{RP}^2$ , leaving only an  $SO(N)$  gauge group for  $N$  D5-branes (instead of the original  $U(N)$ ).

Our description (and that of [6]) of the field theory vacua in terms of supergravity plus D5-branes is only valid when the D3-brane charges  $n_i$  carried by the D5-branes all satisfy  $n_i \gg \sqrt{g_s N} \gg 1$ . However, the matching with the field theory analysis indicates that these vacua make sense in string theory even when this condition is not satisfied. As discussed in [6], in many cases where supergravity breaks down there is an alternative description involving other types of 5-branes, like NS 5-branes. This will be true also for the  $SO(N)$  and  $USp(2N)$  theories. The allowed charges carried by general  $(p, q)$  5-branes wrapped on  $\mathbf{RP}^2$  may be derived by an  $SL(2, \mathbb{Z})$  transformation of the results we described above. For example, a wrapped NS 5-brane will carry an integer 3-brane charge when the RR discrete torsion is non-zero, and the charge will be shifted by a half when there is no RR discrete torsion.

As discussed at the end of section 2, there are also classical field theory vacua which do not correspond to partitions of the type we discussed here, so we cannot map them to vacua including purely D5-branes. It would be interesting to find string theory duals for these vacua.

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